Topic 9: Ordinary Differential Equation

9.1 Basic Concepts and Ideas

Definition:

A ***differential equation*** **(DE)** is an equation involving an unknown function and its derivatives.

Differential equations are classified according to type, order, and linearity.

Classification of differential equation

An equation containing only ordinary derivatives, with respect to a ***single*** *independent variable*, is said to be an ***ordinary*** *differential equation*.

The following are differential equations involving the unknown function *y*.

Example 1:

(i) $\frac{dy}{dx} = \cos x$ or $y' = \cos x$ or $dy = \cos x dx$

(ii) $\frac{dy}{dx} = -\frac{x}{y}$ or $y' = -\frac{x}{y}$ or $dy = -\frac{x}{y}dx$

(iii) $x\frac{dy}{dx} - 4y = x^6 e^x$. or $xy' - 4y = x^6 e^x$

(iv) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$ or $y'' - 5y' + 6y = 0$

A **partial differential equation** (or briefly a **PDE**) is a mathematical equation that involves two or more independent variables, an unknown function (dependent on those variables), and partial derivatives of the unknown function with respect to the independent variables.

Example 2: Here $u = u(t, x)$ is the unknown function with two independent variables *t* and *x*.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{(heat equation)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0 \quad \text{(Laplace's equation)}$$

Classification by Order

The order of the highest-order derivative in a differential equation is called the ***order*** of the equation.

Example 3:

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$ ***second-order*** ordinary differential equation.

$$a^2\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0$$ ***fourth-order*** partial differential equation.

Classification as Linear or Nonlinear

An ordinary differential equation is said to be linear if it can be written in the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

It is characterized by two properties:

(i) The dependent variable y and all its derivatives are of the ***first degree***; that is, the power of each term involving y is 1.

(ii) Each coefficient depends on only the independent variable x.

An equation that is not linear is said to be ***nonlinear***.

Example 4:

$xdy + ydx = 0$ Linear first-order ordinary differential equation

$y'' - 2y' + y = 0$ Linear second-order ordinary differential equation

$x^3\frac{d^3y}{dx^3} - x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 5y = e^x$ Linear third-order ordinary differential equation

$yy'' - 2y' = x$ Nonlinear second-order ordinary differential equation because it involves the product of y and y''.

$\frac{d^3y}{dx^3} - y^2 = 0$ Nonlinear third-order ordinary differential equation

Concept of Solution

Definition: Any function f defined on some interval I, which when substituted into a differential equation reduces the equation to an identity, is said to be a ***solution*** of the equation on the interval.

Example 5:

Verify that $y = x^2$ is a solution of the differential equation (DE) $xy' = 2y$ for all x.

Solution:

To show that $y = x^2$ is a solution of the DE, we have to show that the LHS of the DE is equal to the RHS. Differentiating $y = x^2$ with respect to x *and* substituting $y' = 2x$ into the LHS of the DE ,we obtain

$$\text{LHS} = xy' = x(2x) = 2x^2$$
$$\text{RHS} = 2y = 2x^2$$

We have an identity in x *because LHS=RHS.* Therefore $y = x^2$ is a solution of the DE.

Remark: Verifying that $y = f(x)$ is a solution of a DE is usually relatively easy as it involves differentiation. Solving a DE is much more difficult as it involves finding the unknown function $y = f(x)$.

Explicit and Implicit Solutions

A solution of an ordinary differential equation that can be written in the form $y = f(x)$ is said to be an ***explicit solution***. It is also a solution in which the dependent variable is expressed solely in terms of the independent variable and constant.

A relation $\mathbf{G}(x, y) = 0$ is said to be an ***implicit solution*** of an ordinary differential equation on an interval I provided it determines implicitly a differentiable function $y = f(x)$ that satisfies the differential equation on I.

Example 6:

For $-1 < x < 1$, show that the relation $x^2 + y^2 - 1 = 0$ is an implicit solution of the differential equation $\frac{dy}{dx} = -\frac{x}{y}$.

Solution: We are going to show by differentiating $x^2 + y^2 - 1 = 0$ with respect to x, we arrive at the DE $\frac{dy}{dx} = -\frac{x}{y}$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) - \frac{d}{dx}(1) = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Example 7:

Show that the function $y = 3xe^x$ is a solution of the linear (differential) equation

$$y'' - 2y' + y = 0$$

Solution: We find $y' = 3xe^x + 3e^x$

$$y'' = 3xe^x + 3e^x + 3e^x$$
$$= 3xe^x + 6e^x$$

Therefore

$$y'' - 2y' + y = (3xe^x + 6e^x) - 2(3xe^x + 3e^x) + 3xe^x = 0$$

Hence $y = 3xe^x$ is a solution of the DE

In general, it can be shown that $y = Axe^x$, where A is an arbitrary constant, is a solution of the differential equation $y'' - 2y' + y = 0$.

Hence this is known as the ***general solution*** of the differential equation while $y = 3xe^x$ is a ***particular solution***.

The most general function that will satisfy the differential equation contains one or more arbitrary constants; it is known as the ***general solution*** of the differential equation. Giving particular numerical values to one or more of the constants in the general solution results in a ***particular solution*** of the equation.

Example 8:

Solve $y' = \cos x$.

Solution:

$$y = \sin x + c \quad \text{with arbitrary } c.$$

Figure 1 shows some of the solutions, for c = -3, -2, -1, 0, 1, 2, 3, 4.

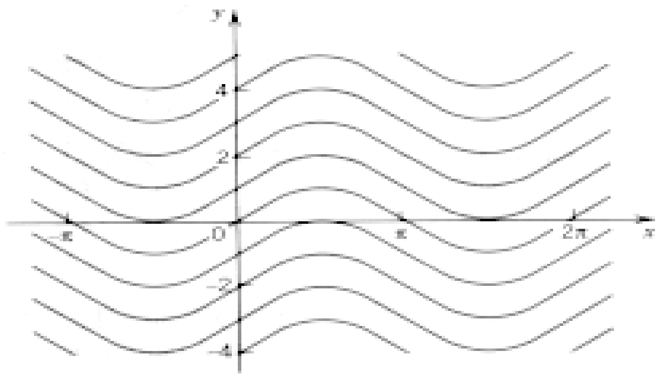

Fig. 1. Solutions of $y' = \cos x$

<u>Initial-Value Problem</u>

An **initial value problem** is an ordinary differential equation

$$F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^n y}{dx^n}\right) = 0$$ (which is an ***nth-order*** differential equation)

together with the initial condition

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots, \quad y^{(n-1)}(x_0) = y_{n-1},$$

where $x_0, y_0, y_1, \ldots, y_{n-1}$ are arbitrary constants.

Example 9 :

1. The initial value problem. $y'(x) = y$; $y(0) = 3$
2. The initial value problem $\dfrac{d^2y}{dx^2} + y = 0;\ y(0) = -1,\ y'(0) = 1.$

9.2 Separable Differential Equations

Definition:

A first-order differential equation that can be expressed in the form

$$g(y)\frac{dy}{dx}=f(x) \quad \text{or} \quad g(y)dy=f(x)dx \qquad (1)$$

is said to be ***separable*** or to have ***separable variables*** where *f(x)* is a function that depends only on *x* and *g(y)* is a function that depends only on *y*.

Example 10: Show that $\frac{dy}{dx}=xe^{(x+2y)}$ is separable.

Solution:

$$\frac{dy}{dx}=xe^{x}e^{2y}\ ; \qquad dy=xe^{x}e^{2y}dx\ ;$$

$$e^{-2y}dy=xe^{x}dx$$

which is of the form $g(y)dy=f(x)dx$

Example 10a: The differential equation $\frac{dy}{dx}=3x-y$ is not separable because it cannot be expressed in the form $g(y)dy=f(x)dx$

Method of Solution : Separable equation

To solve a separable DE $g(y)\frac{dy}{dx}=f(x)$ we integrate on both sides with respect to *x*, obtaining

$$\int g(y)\frac{dy}{dx}dx=\int f(x)dx+c.$$

$$\int g(y)dy=\int f(x)dx+c.$$

Example 11:

Solve the differential equation $\frac{dy}{dx}=1+y$

Solution: We note that the DE is separable because it can be expressed in the form $g(y)dy=f(x)dx$

$$\frac{1}{1+y}dy=dx$$

$$\int\frac{1}{1+y}dy=\int dx$$

$$\ln|1+y|=x+c$$

This is an implicit solution of the DE. It can be converted into an explicit solution of the form $y=f(x)$. How?

Example 12:

Solve the differential equation $9yy' + 4x = 0$.

Solution:

$$\frac{dy}{dx} = \frac{-4x}{9y}$$

$$\int 9y\,dy = -\int 4x\,dx$$

$$\frac{9}{2}y^2 = -2x^2 + c^*$$

$$\frac{x^2}{9} + \frac{y^2}{4} = c$$

The solution represents a family of ellipses.

9.3 Linear Differential Equations

Definition:

A differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$ is said to be a ***first-order linear equation***.

For example,

$$x\frac{dy}{dx} - 4y = x^6e^x$$

is a first order linear DE.

Here $a_1(x) = x, \quad a_0(x) = -4, \quad and \quad g(x) = x^6e^x$

Method of solution :First Order Linear Differential equation

1. Make the coefficient of $\frac{dy}{dx}$ unity. i.e.

$$\frac{dy}{dx} + P(x)y = r(x)$$

For homogeneous equation, $r(x) = 0$,

$$\frac{dy}{dx} + P(x)y = 0$$ is a separable equation.

2. Identify $p(x)$ and find the integrating factor

$$\mu(x) = e^{\int P(x)dx}.$$

3. Multiply the equation obtained in step (1) by the integrating factor:

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = e^{\int P(x)dx}r(x).$$

4. The left side of the equation in step (3) is the derivative of the product of the integrating factor and the dependent variable y; that is,

$$\frac{d}{dx}[e^{\int P(x)dx}y] = e^{\int P(x)dx}r(x).$$

5. Integrate both sides of the equation found in step (4).

Example 13:

Solve $x\dfrac{dy}{dx} - 4y = x^6 e^x$.

Solution:

1. Rewrite the DE as $\dfrac{dy}{dx} - \dfrac{4}{x}y = x^5 e^x$.

2 We then note that $P(x) = -\dfrac{4}{x}$. Hence, the integrating factor is given by

$$\mu(x) = e^{\int\left(-\frac{4}{x}\right)dx} = e^{-4\ln x} = e^{\ln(x^{-4})} = \frac{1}{x^4} \quad \text{because} \quad e^{\ln f(x)} = f(x)$$

3. $\therefore \dfrac{1}{x^4}\dfrac{dy}{dx} - \dfrac{4}{x^5}y = \dfrac{1}{x^4}(x^5 e^x)$

4. $\dfrac{d}{dx}\left(\dfrac{1}{x^4}y\right) = xe^x$

5. $\dfrac{1}{x^4}y = \int xe^x dx = xe^x - e^x + c$

$$y = x^5 e^x - x^4 e^x + cx^4$$

Example 14:

Solve the initial value problem: *y'* + 2*xy* = *x*, *y*(0) = 1.

Solution:

Here $P(x) = 2x$,

Integrating factor, $\mu(x) = e^{\int P(x)dx} = e^{\int 2x dx} = e^{x^2}$.

Multiplying into the equation, $e^{x^2}\left(\dfrac{dy}{dx} + 2xy\right) = xe^{x^2}$

$$\frac{d}{dx}\left(e^{x^2}y\right) = xe^{x^2}$$

$$e^{x^2}y = \int xe^{x^2}dx = \frac{1}{2}e^{x^2} + c$$

$$\therefore \quad y(x) = \frac{1}{2} + ce^{-x^2}.$$

From the initial condition, when $x = 0$, $y = 1$

$\therefore \quad 1 = \dfrac{1}{2} + c$ Hence, $c = \dfrac{1}{2}$

The solution of our initial value problem is $y(x) = \dfrac{1}{2} + \dfrac{1}{2}e^{-x^2}$.

Tri1-1617

9.4 Exact Differential Equations

Revision on Partial Differentiation (Topic 8)

Example:

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ if $f(x,y) = x^2 + 3xy + y - 1$.

Solution: Regarding y as a constant and differentiating f(x,y) with respect to x, we obtain

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + 3xy + y - 1\right) = 2x + 3y$$

Regarding x as a constant and differentiating f(x,y) with respect to y, we obtain

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + 3xy + y - 1\right) = 3x + 1$$

Example:

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ if $f(x,y) = y\sin xy$

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(y\sin xy) = y\frac{\partial}{\partial x}(\sin xy) = y^2\cos xy$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(y\sin xy\right) = y\frac{\partial}{\partial y}\left(\sin xy\right) + (\sin xy)\frac{\partial}{\partial y}(y)$$

$$= y\cos xy\frac{\partial}{\partial y}(xy) + \sin xy = xy\cos xy + \sin xy$$

Definition of Total Differential

If $f = f(x,y)$ then the differential of f, denoted df, is defined by

$$df = f_x(x,y)dx + f_y(x,y)dy \quad \text{or} \quad df = \frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy$$

df is also called the total differential of f.

Example: Let $F = F(x,y) = \frac{1}{3}x^3y^3$. Then

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = \frac{\partial}{\partial x}(\frac{1}{3}x^3y^3)dx + \frac{\partial}{\partial y}(\frac{1}{3}x^3y^3)dy$$

$$dF = x^2y^3dx + x^3y^2dy$$

Definition of Exact Differential Equations

A differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is said to be ***exact*** in a region ***R*** of the *xy*-plane if there is a function *F(x, y)* such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial F(x,y)}{\partial y} = N(x,y).$$

That is, the total differential of *F* satisfies

$$dF(x,y) = M(x,y)dx + N(x,y)dy.$$

Example 15:

1. Show that the differential equation $x^2y^3dx + x^3y^2dy = 0$ is exact.

Solution: To show that the DE is exact we have to find a function $F(x,y)$ such that its differential

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = x^2y^3dx + x^3y^2dy$$

We claim that $F(x,y) = \frac{1}{3}x^3y^3$ is such a function because $\frac{\partial F}{\partial x} = x^2y^3$ and $\frac{\partial F}{\partial y} = x^3y^2$

$$dF = x^2y^3dx + x^3y^2dy.$$

Remark : In practice, producing such a function $F(x,y)$ to show that the DE is exact is not that easy. In fact if we can produce such a function, then the solution of the DE is given implicitly by $F(x,y) = c$. Later we will give an easier criterion for testing whether a given DE is exact or not.

Example 15a: Solve $\frac{dy}{dx} = \frac{\sin y}{2y - x\cos y}$.

Solution: The above d.e. in differential form can be written as

$$\sin y\, dx + (x\cos y - 2y)dy = 0$$

To solve the DE we would have to produce a function $F(x,y)$ such that the LHS of the above DE is $dF(x,y)$, *the total differential of* $F(x,y)$. We can verify that such a function is

$$F(x,y) = x\sin y - y^2.$$

Therefore

$$d(x\sin y - y^2) = 0$$

Hence $x\sin y - y^2 = c$ *is the solution of the DE.*

Trial-1617

Theorem (Criterion for an Exact Differential)

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R. Then a necessary and sufficient condition that

$$M(x, y)dx + N(x, y)dy$$

be an exact differential is $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Method of solution : Exact equation

1. If $Mdx + Ndy = 0$ is exact, then $\dfrac{\partial F}{\partial x} = M$. Integrate this last equation with respect to x to get $F(x, y) = \int M(x, y)dx + g(y)$. (2)

2. To determine $g(y)$, take the partial derivative with respect to y of both sides of equation (2) and substitute N for $\dfrac{\partial F}{\partial y}$. We can now solve for $g'(y)$.

3. Integrate $g'(y)$ to obtain $g(y)$ up to a numerical constant. Substituting $g(y)$ into equation (2) gives $F(x, y)$.

4. The solution to $Mdx + Ndy = 0$ is given implicitly by $F(x, y) = C$.
(Alternatively, starting with $\dfrac{\partial F}{\partial y} = N$, the implicit solution can be found by first integrating with respect to y)

Example 16:

Solve $(e^{2y} - y\cos xy)dx + (2xe^{2y} - x\cos xy + 2y)dy = 0$.

Solution:

Here $M(x, y) = (e^{2y} - y\cos xy)$ and $N(x, y) = (2xe^{2y} - x\cos xy + 2y)$.

Therefore $\dfrac{\partial M}{\partial y} = 2e^{2y} + xy\sin xy - \cos xy = \dfrac{\partial N}{\partial x}$, the equation is exact.

Hence, a function $F(x, y)$ exists for which $M(x, y) = \dfrac{\partial F}{\partial x}$ and $N(x, y) = \dfrac{\partial F}{\partial y}$.

$$\therefore \frac{\partial F}{\partial x} = e^{2y} - y\cos xy$$

$$F(x, y) = \int e^{2y}dx - y\int \cos xy dx = xe^{2y} - \sin xy + g(y)$$

$$\frac{\partial F}{\partial y} = 2xe^{2y} - x\cos xy + g'(y) = N = 2xe^{2y} - x\cos xy + 2y$$

so that $g'(y) = 2y$ and $g(y) = y^2 + c$.

Hence, a one parameter family of solutions is given by

$$xe^{2y} - \sin xy + y^2 + C = 0.$$

THE END
(nby, July 2016)

Topic 9b: Second Order Differential Equations

9.5 SOLVING SECOND ORDER DIFFERENTIAL EQUATIONS

A second-order differential equation is called ***linear*** if it can be written as

$$y'' + p(x)y' + q(x)y = r(x) \qquad (1)$$

where ***p, q, r*** are any given function of ***x***. Any *second* order differential equation that cannot be written in the above form is called ***nonlinear***.

If ***r(x) = 0***, equation (1) becomes

$$y'' + p(x)y' + q(x)y = 0 \qquad (2)$$

and is called ***homogeneous***.

If ***r(x)*** is not identically zero, the equation is called ***non-homogeneous***.

Example 1

$y'' + 4y = e^{-x}\sin x$ --- *non-homogeneous linear d e*

$(1 - x^2)y'' - 2xy' + 6y = 0$ --- *homogeneous linear d.e*

$x(y''y + y'^2) + 2y'y = 0$ --- *homogeneous nonlinear d.e*

Theorem (*Fundamental theorem for the homogeneous equation*)
For a homogeneous linear differential equation (2), any linear combination of two solutions on an open interval *I* is again a solution of (2) on *I*. In particular, for such an equation, sums and constant multiples of solutions are again solutions.

Example 2

1. Verify that $y = e^x$ and $y = e^{-x}$ are solutions of the homogeneous linear differential equation

$$y'' - y = 0$$

2. Are $y = ce^x$, $y = de^{-x}$ and $y = ce^x + de^{-x}$ also solutions?

Solution:

1.

When $y = e^x$, $y' = e^x$ and $y'' = e^x$ Hence $y'' - y = e^x - e^x = 0$ Therefore, $y = e^x$ is a solution for the d.e.	When $y = e^{-x}$, $y' = -e^{-x}$ and $y'' = e^{-x}$ Hence $y'' - y = e^{-x} - e^{-x} = 0$ Therefore, $y = e^{-x}$ is also a solution for the d.e.

2.

When $y = ce^x$, $y' = ce^x$ and $y'' = ce^x$ Hence $y'' - y = ce^x - ce^x = 0$ Therefore, $y = ce^x$ is a solution for the d.e.	When $y = de^{-x}$, $y' = -de^{-x}$ and $y'' = de^{-x}$ Hence $y'' - y = de^{-x} - de^{-x} = 0$ Therefore, $y = de^{-x}$ is also a solution for the d.e.

Similarly,

$$y = ce^x + de^{-x}$$
$$y' = ce^x - de^{-x}$$
$$y'' = ce^x + de^{-x}$$
$$\therefore y'' - y = (ce^x + de^{-x}) - (ce^x + de^{-x}) = 0$$

Therefore, $y = ce^x + de^{-x}$ is another solution for the d.e.

Note*: This theorem does not hold for the non-homogeneous equation or for a nonlinear equation.*

Tri 1-16/17

General Solution

For second-order homogeneous linear equations (2), a ***general solution*** will be of the form

$$y = c_1 y_1 + c_2 y_2 \qquad (3)$$

a linear combination of two (suitable) solutions involving two arbitrary constants c_1, c_2. These two solutions (y_1 and y_2) form a ***basis*** (or ***fundamental set***) of solutions to the d.e. (2) on I.

Particular Solution

A ***particular solution*** of (2) on I is obtained if we assign specific values to c_1 and c_2 in (3).

Initial Value Problem

For second-order homogeneous linear equations, an ***initial value problem*** would consist of a homogeneous linear differential equation $y'' + p(x)y' + q(x)y = 0$
and two initial conditions $y(x_0) = K_0$, $y'(x_0) = K_1$,

Linear independence and dependence

Two functions $\boldsymbol{y_1}(x)$, $\boldsymbol{y_2}(x)$ are said to be linearly dependent on an interval $\boldsymbol{I}$ if there exist constants c_1, c_2 not all zero, such that

$$\boldsymbol{c_1 y_1(x) + c_2 y_2(x) = 0}$$

for every x in the interval.

It is said to be ***linearly independent*** on an interval $\boldsymbol{I}$ if it is not linearly dependent on the interval.

Example 3

The function $f_1(x) = \sin 2x$ and $f_2(x) = \sin x \cos x$ are linearly dependent on the interval $(-\infty, \infty)$ since

$$c_1 \sin 2x + c_2 \sin x \cos x = 0$$

is satisfied for every real x if we choose $c_1 = \frac{1}{2}$ and $c_2 = -1$.

Definition of a basis

A basis of solutions of (2) on an interval I is a pair y_1, y_2 of ***linearly independent*** solutions of (2) on I.

9.5.1 HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

In this section, we show how to solve homogeneous second order linear equations

$$ay'' + by' + cy = 0 \qquad (4)$$

where the coefficients $a(\neq 0)$, b and c are constants.

We try a solution of the form $y = e^{\lambda x}$. Then $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$.
Equation (4) becomes

$$a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0$$
$$(a\lambda^2 + b\lambda + c)e^{\lambda x} = 0.$$

Because $e^{\lambda x}$ is never zero for real values of x,

Tri 1 1617

$$a\lambda^2 + b\lambda + c = 0.$$

This latter equation is called the ***auxiliary equation***, or ***characteristic equation***.

The roots of the auxiliary equation are

$$\lambda_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}, \qquad \lambda_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

With that, we obtain

Case I: two real roots if $b^2 - 4ac > 0$
Case II: a real double root if $b^2 - 4ac = 0$
Case III: complex conjugate roots if $b^2 - 4ac < 0$

Consider these three cases, namely, the solutions of the auxiliary equation corresponding to distinct real roots, real but equal roots, and a conjugate pair of complex roots.

CASE 1: DISTINCT REAL ROOTS ($\lambda_1 \neq \lambda_2$)

The general solution of (4) on ***R*** is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

where c_1 and c_2 are arbitrary constants.

Example 4

Find the general solution of $y'' + 5y' + 6y = 0.$

Solution:

The characteristic equation is

$\lambda^2 + 5\lambda + 6 = 0$

$(\lambda + 2)(\lambda + 3) = 0$

$\lambda = -2$ or $\lambda = -3.$ The roots are -2 and -3.

Thus, the general solution is $y = c_1 e^{-2x} + c_2 e^{-3x}$.

CASE II: REPEATED REAL ROOTS ($\lambda_1 = \lambda_2$)

The general solution of (4) on ***R*** is

$$y = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$$

where c_1 and c_2 are arbitrary constants.

Example 5

Solve the differential equation $y'' + 4y' + 4y = 0.$

Solution:

The characteristic equation is

$\lambda^2 + 4\lambda + 4 = 0$

$(\lambda + 2)^2 = 0$ So $\lambda = -2$ (repeated)

Thus, the general solution is $y = c_1 e^{-2x} + c_2 x e^{-2x}$.

Tri1-1617

CASE III: CONJUGATE COMPLEX ROOTS (λ_1, λ_2 are complex)

If λ_1 and λ_2 are complex, then we can write

$$\lambda_1 = \alpha + i\beta \text{ and } \lambda_2 = \alpha - i\beta$$

where α and $\beta > 0$ are real.

Therefore, the general solution of (4) on ***R*** is

$$y = Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x}$$

which can be expressed in the following form by using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$

$$y = c_1e^{\alpha x}\cos\beta x + c_2e^{\alpha x}\sin\beta x$$
$$= e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x).$$

where c_1 and c_2 are arbitrary constants.

Example 6

Find the general solution of $y'' + 9y = 0$.

Solution:

The characteristic equation is

$$\lambda^2 + 9 = 0$$
$$\lambda = \pm 3i$$

The general solution is $y = c_1\cos 3x + c_2\sin 3x$.

Summary of Case I, II, and III

$ay'' + by' + cy = 0$ …….. (4)

Case	Roots of ***characteristic equation*** $a\lambda^2 + b\lambda + c = 0$	Basis of solutions of (4)	General Solution of (4)
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$
II	Repeated real root $\lambda = \lambda_1 = \lambda_2$	$e^{\lambda x}, xe^{\lambda x}$	$y = (c_1 + c_2x)e^{\lambda x}$
III	Complex conjugates $\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$	$e^{\alpha x}\cos\beta x$, $e^{\alpha x}\sin\beta x$	$y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$ or $y = c_1e^{\alpha x}\cos\beta x + c_2e^{\alpha x}\sin\beta x$

Tri1-1617

9.5.2 NON-HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

In this section, we show how to solve non-homogeneous linear differential equations

$$a\,y'' + by' + cy = r(x) \qquad (5)$$

where $a, b,$ *and* c are constants and $r(x) \neq 0$.

The corresponding homogeneous equation of (5) is

$$ay'' + by' + cy = 0 \qquad (6)$$

It can be shown that the ***general solution*** of the non-homogeneous equation (5) is given by

$$y = y_h(x) + y_p(x) \qquad (7)$$

where $y_h = c_1y_1(x) + c_2y_2(x)$ (also known as ***complementary function***) is the general solution of the homogeneous equation (6) and y_p is a ***particular solution*** of (5).

Method of solving nonhomogeneous DE with constant coefficients	
Step 1:	Solve for homogeneous equation (6).
Step 2:	Find any particular solution y_p of (5).
Step 3:	Form general solution $y = y_h + y_p$

Example 7

Find a particular solution of $y'' + 9y = 27$.

Solution: Since $r(x) = 27$ we assume that a particular solution is given by $y_p = A$ where A is a constant. Substituting $y_p = A$ into the above DE and noting that $y_p'' = 0$, we have

$$y_p'' + 9\,y_p = 0 + 9\,A = 27.$$

Therefore $A = 3$ *and a particular solution is given by* $y_p = 3$.

9.4.2.1 Method of Undetermined coefficients

The method of undetermined coefficient is a technique for determining a particular solution y_p.

Rules for the Method of Undetermined Coefficients

(a) Basic Rule.

If $r(x)$ is one of the functions in the first column in the table below, choose the corresponding function y_p in the second column and determine its undetermined coefficients by substituting y_p and its derivatives into (5).

Term in $r(x)$	Choice for y_p
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \, (n = 0, 1, \cdots)$	$K_n x^n + K_{n-1} x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$ $k \sin \omega x$	$K \cos \omega x + M \sin \omega x$
$ke^{\alpha x} \cos \omega x$ $ke^{\alpha x} \sin \omega x$	$e^{\alpha x}\left(K \cos \omega x + M \sin \omega x\right)$
$x^n \cos \omega x$ $x^n \sin \omega x$	$(K_n x^n + K_{n-1} x^{n-1} + \cdots + K_0)\cos \omega x + (L_n x^n + L_{n-1} x^{n-1} + \cdots + L_0)\sin \omega x$

Example 8

Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6.$

Solution:

Step 1. ***We first solve the associated homogeneous equation***

$$y'' + 4y' - 2y = 0.$$

The characteristic equation is

$$\lambda^2 + 4\lambda - 2 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16+8}}{2} = -2 \pm \sqrt{6}$$

$$\therefore y_h = c_1 e^{(-2+\sqrt{6})x} + c_2 e^{(-2-\sqrt{6})x}$$

Step 2. ***Solve for particular solution.***

Since $r(x) = 2x^2 - 3x + 6$ is a quadratic polynomial, we assume

$$y_p = Ax^2 + Bx + C.$$

Then $y_p' = 2Ax + B$ and $y_p'' = 2A.$

Substituting into the equation, we have

$$2A + 4(2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 - 3x + 6$$

Equating coefficients: $-2A = 2,\ 8A - 2B = -3,\ 2A + 4B - 2C = 6$

Solving: $A = -1,\ B = -\frac{5}{2},\ C = -9$

$$\therefore y_P = -x^2 - \frac{5}{2}x - 9$$

Step 3. ***The general solution of the given equation is***

$$y(x) = y_h + y_p = c_1 e^{(-2+\sqrt{6})x} + c_2 e^{(-2-\sqrt{6})x} - x^2 - \tfrac{5}{2}x - 9$$

(b) Sum Rule.

If $r(x)$ consists of sum of m terms of the kind given in above table, the assumption for a particular solution of y_p consists of the sum of the trial forms $y_{p_1}, y_{p_2}, \cdots, y_{p_m}$ corresponding to these terms

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_m}.$$

Tri1-1617

Example 9

Find the general solution of the equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = e^{-2x} + 2 - x.$$

Solution:

Step 1. ***We first solve the associated homogeneous equation***

The characteristic equation is

$$\lambda^2 + 5\lambda - 6 = 0$$
$$(\lambda - 1)(\lambda + 6) = 0$$
$$\lambda = 1 \text{ or } \lambda = -6$$
$$\therefore y_h = c_1e^x + c_2 e^{-6x}$$

Step 2. ***Solve for particular solution.***

Since $r(x) = e^{-2x} + 2 - x$ is the sum of two types of functions from the table in (a) (viz. exponential + polynomial) , we assume

$$y_{p_1} = Ae^{-2x},\ y_{p_2} = Bx + C$$

Let $y_p = Ae^{-2x} + Bx + C$

$$\therefore \quad y_p' = -2Ae^{-2x} + B$$
$$y_p'' = 4Ae^{-2x}$$

Substituting into the equation, we have

[*You are required to fill in the intermediate steps.*]

$$-12A = 1, \qquad -6B = -1, \qquad 5B - 6C = 2$$

$$A = -\frac{1}{12}, B = \frac{1}{6}, C = -\frac{7}{36}$$

$$\therefore y_P = \ldots$$

Step 3. ***The general solution of the given equation is***

$$y = y_h + y_p = c_1e^x + c_2e^{-6x} - \frac{e^{-2x}}{12} + \frac{x}{6} - \frac{7}{36}$$

(c) Modification Rule.

If a term in your choice for y_{p_i} contains terms that duplicate terms in y_h, then that y_{p_i} must be multiplied by x^n, where n is the smallest positive integer that eliminates that duplication.

Example 10

Find the general solution of the equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = e^t$$

Solution:

Step 1. ***We first solve the associated homogeneous equation***

The characteristic equation is

$$\lambda^2 - 2\lambda + 1 = 0$$

$(\lambda - 1)^2 = 1$ [*You are required to fill in the intermediate steps.*]

$\therefore y_h = c_1e^t + c_2\, te^t$

Step 2. ***Solve for particular solution.***

Since $r(t) = e^t$ is a term in y_c, we assume

$$y_p = At^2e^t$$

$\therefore$ $y_p' = 2Ate^t + At^2e^t$

$y_p'' = 2Ae^t + 4Ate^t + At^2e^t$

[*You are required to fill in the intermediate steps.*]

Substituting into the equation, we have $A = \frac{1}{2}$

Step 3. ***The general solution of the given equation is***

$$y = y_h + y_p = c_1e^t + c_2te^t + \frac{1}{2}t^2e^t.$$

Example 11

Given that the function $y_1(x) = e^{-5x}$ and $y_2(x) = e^{2x}$ are both the solutions of the homogeneous equation, find the general solution of the equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 10y = x(e^x + 1)$$

Solution:

Step 1. ***We first determine the solution of the associated homogeneous equation***

Since $y_1(x) = e^{-5x}$ and $y_2(x) = e^{2x}$ are both the solutions of the homogeneous equation

$$\therefore y_h = c_1e^{-5x} + c_2e^{2x}$$

Step 2. ***Solve for particular solution.***

Since $r(x) = x(e^x + 1)$ is a combination of two functions, we assume

$y_p = (Ax + B)e^x + Cx + D$ [*Do you understand how the rules are applied?*]

$\therefore$ $y_p' = (Ax + B)e^x + Ae^x + C$

$y_p'' = (Ax + B)e^x + 2Ae^x$

[*You are required to fill in the intermediate steps.*]

Substituting into the equation, we have

$$A = -\frac{1}{6} \qquad B = -\frac{5}{36} \qquad C = -\frac{1}{10} \qquad D = -\frac{3}{100}$$

Step 3. ***The general solution of the given equation is***

$$y = y_h + y_p = c_1e^{-5x} + c_2e^{2x} + \left(-\frac{1}{6}x - \frac{5}{36}\right)e^x - \frac{1}{10}x - \frac{3}{100}$$

---------------------------THE END----------------------------

(nby, July 2016)

Trili-1617